Invariant Manifolds for Stochastic $2D$ Navier-Stokes Equations

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\begin{aligned}
du - \nu \triangle u\, dt + (u \cdot \nabla) u\, dt + \nabla p\, dt \\
= \gamma u\, dt + \sigma_0\, dW_0(t, x) + \sum_{k=1}^{\infty} \sigma_k u(t)\, dW_k(t) \\
(\nabla \cdot u)(t, x) = 0, \quad x \in D,\ t > 0, \\
u(t, x) = 0, \quad x \in \partial D,\ t > 0, \\
u(0, x) = f(x), \quad x \in D.
\end{aligned}
\tag{1}
$$

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$\sigma_k \in L(\mathbf{R}^2),\ k \geq 1,$ commuting, symmetric (2×2)-matrices- $\sum_{k=1}^{\infty} |\sigma_k|^2 < \infty,\ |\sigma_k|^2 := \mathrm{tr}(\sigma_k^2)$;

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initial velocity $f : D \to \mathbf{R}^2$.

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Perfection:
A family of propositions $\{P(\omega) : \omega \in \Omega\}$ holds perfectly in ω if there is a sure event $\Omega^* \in \mathcal{F}$ such that $\theta(t, \cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbf{R}$ and $P(\omega)$ is true for every $\omega \in \Omega^*$.

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Regularity:

$C^{1,1}$ = Fréchet differentiable with derivatives Lipschitz on bounded sets.

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To establish:

- existence of a perfect locally compacting $C^{1,1}$ cocycle (semiflow) generated by all solutions of the stochastic Navier-Stokes equation;
- large-time asymptotics for the linearized stochastic semiflow on a stationary solution, given by a countable non-random Lyapunov spectrum of the cocycle;
- existence of flow-invariant C^1 local stable/unstable manifolds in the neighborhood of a hyperbolic stationary solution;

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- existence of a countable, global invariant flag relative to an ergodic stationary point (when $\gamma = 0$);
- sufficient conditions on the parameters ν, γ, $\sigma_i, i \geq 1$, (with $\sigma_0 = 0$) and the geometry of the domain D to guarantee uniqueness and hyperbolicity of the stationary solution (viz. the zero equilibrium).

The set-up

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Consider the Hilbert space

$$V := \{v \in H_0^1(D, \mathbf{R}^2) : \nabla \cdot v = 0 \; a.e. \text{ in } D\},$$

with the norm

$$||v||_V := \Big(\int_D |\nabla v(x)|^2 \, dx \Big)^{\frac{1}{2}}$$

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$H :=$ closure of V in the L^2-norm

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The (Stokes) operator A in H is given by

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Define the bilinear operator B by

$$B(u, v) := P_H\big((u \cdot \nabla)v\big),$$

whenever u, v are such that $(u \cdot \nabla v)$ belongs to L^2.

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Apply the projection P_H to each term of the SNSE (1) and get abstract form:

$$\left.\begin{aligned} du(t)+Au(t)\,dt + B(u(t))\,dt \\ = \gamma u(t)\,dt + \sigma_0\,dW_0(t) + \sum_{k=1}^{\infty} \sigma_k^H u(t)\,dW_k(t) \\ u(0) = f \in H \end{aligned}\right\} \tag{2}$$

in $L^2(0,T;V')$; $V' :=$ dual of V;
$\sigma_k^H f := P_H(\sigma_k \circ f),\ f \in H.$

Existence of the Cocycle

We show that strong solutions of the SNSE generate a Fréchet $C^{1,1}$ locally compacting cocycle (viz. stochastic semiflow) $u : \mathbf{R}^+ \times H \times \Omega \to H$ on the Hilbert space H.

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We show that strong solutions of the SNSE generate a Fréchet $C^{1,1}$ locally compacting cocycle (viz. stochastic semiflow) $u : \mathbf{R}^+ \times H \times \Omega \to H$ on the Hilbert space H.

Use a variational technique which transforms the SNSE into a *random* NSE. Then analyze the random NSE via a priori estimates coupled with lengthy estimates on Galerkin approximations. (cf. [Te], [Ro]).

Existence of the Cocycle-contd

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du(t,f) + Au(t,f)dt + B(u(t,f))dt \\
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For each $f \in H$, the SNSE (3) admits a unique strong solution

$$u(\cdot, f) \in L^2(\Omega; C([0,T]; H)) \cap L^2(\Omega \times [0,T]; V)$$

([B-C-F]).

The Cocycle: Theorem

Let $u(t, f, \cdot)$ be the unique global solution of the SNSE (3) for $t \geq 0$ and $f \in H$. Denote by $\theta : \mathbf{R}^+ \times \Omega \to \Omega$ the standard Brownian shift

$$\theta(t, \omega)(s) := \omega(t+s) - \omega(t), \quad t, s \geq 0, \ \omega \in \Omega, \quad \textbf{(4)}$$

on Wiener space $(\Omega, \mathcal{F}, P)$.

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Then there is a version $u : \mathbf{R}^+ \times H \times \Omega \to H$ of the solution of (3) with the following properties:

The Cocycle: Theorem-contd

- The map $u : \mathbf{R}^+ \times H \times \Omega \longrightarrow H$ is jointly measurable, and for each $f \in H$, the process $u(\cdot, f, \cdot) : \mathbf{R}^+ \times \Omega \longrightarrow H$ is $(\mathcal{F}_t)_{t \geq 0}$-adapted.

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- For each $t > 0$ and $\omega \in \Omega$, the map $u(t, \cdot, \omega) : H \to H$ takes bounded sets into relatively compact sets.

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- For each $t > 0$ and $\omega \in \Omega$, the map $u(t, \cdot, \omega) : H \to H$ takes bounded sets into relatively compact sets.
- (u, θ) is a $C^{1,1}$ perfect cocycle; viz.

$$u(t_2, u\big(t_1, f, \omega), \theta(t_1, \omega)) = u(t_1 + t_2, f, \omega) \tag{5}$$

for all $t_1, t_2 \geq 0$, $f \in H$, $\omega \in \Omega$.

The Cocycle: Theorem-contd

- For each $(t, f, \omega) \in \mathbf{R}^+ \times H \times \Omega$, the Fréchet derivative $Du(t, f, \omega) \in L\big(H\big)$ of the map $u(t, \cdot, \omega)$ is compact linear, and the map

$$\begin{aligned} \mathbf{R}^+ \times H \times \Omega &\longrightarrow L\big(H\big) \\ (t, f, \omega) &\longmapsto Du(t, f, \omega) \end{aligned}$$

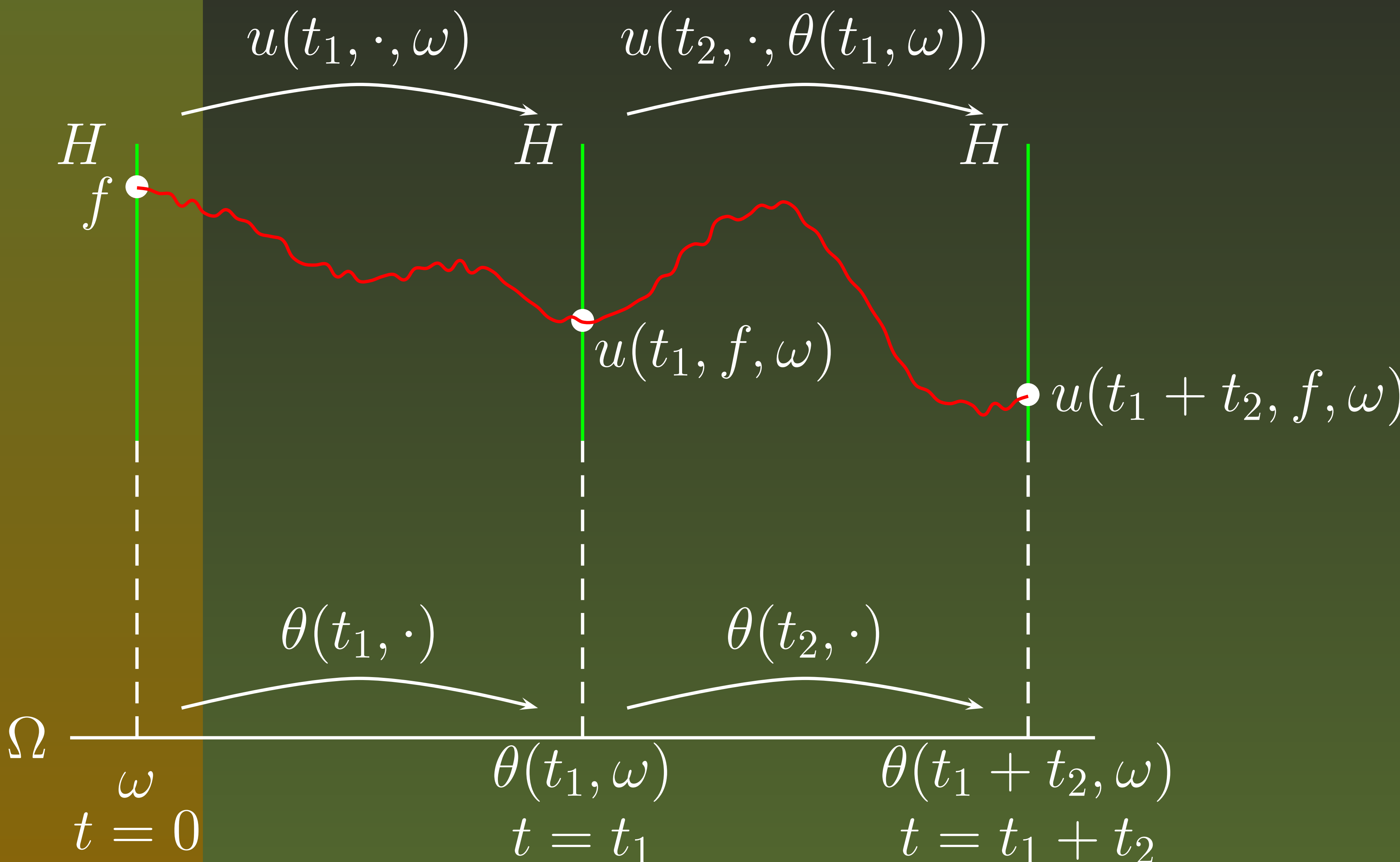
is strongly measurable.

The Cocycle: Theorem-contd

- For fixed $\rho, a > 0$,

$$E \log^+ \sup_{\substack{0 \le t_1, t_2 \le a \\ |f|_H \le \rho}} \Big\{ |u(t_2, f, \theta(t_1, \cdot))|_H + \|Du(t_2, f, \theta(t_1, \cdot))\|_{L(H)} \Big\} < \infty. \tag{6}$$

The cocycle property



Proof of Theorem: Sketch

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Define $u : \mathbf{R}^+ \times H \times \Omega \to H$ by setting

$$u(t, f, \omega) := Q(t, \omega)[v(t, f, \omega) + Z(t, \omega)], \qquad \textbf{(7)}$$

for $t \geq 0,\ \omega \in \Omega, f \in H$;

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for $t \geq 0$, $\omega \in \Omega, f \in H$; $Q : [0, \infty) \times \Omega \to L(\mathbf{R}^2)$ satisfies

$$Q(t) = I + \gamma \int_0^t Q(s)\, ds + \sum_{k=1}^{\infty} \int_0^t \sigma_k Q(s)\, dW_k(s),\ t \geq 0$$

(8)

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$$Z(t) := \sigma_0 \int_0^t Q(s)^{-1} T_{t-s}\, dW_0(s), \quad t \geq 0;$$

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$T_t := \exp(-tA) : H \to H$, $t \geq 0$, semigroup of $-A$.

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Let $v(t) \equiv v(t, f)$ satisfy the random NSE:

$$
\begin{aligned}
dv(t) &= -Av(t)\,dt \\
&\qquad - Q(t)B\Big(Q(t)(v(t) + Z(t)), v(t) + Z(t)\Big)\,dt, \\
v(0) &= f \in H\,.
\end{aligned}
\tag{9}
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Existence of a unique global solution to the random NSE (9) follows by Galerkin approximations, a priori estimates and compactness of the embedding $V \to H$. Obtain Lipschitz and Fréchet differentiability ($C^{1,1}$) properties for v and hence for u using very lengthy estimates on v and its Gateaux derivatives.

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To show the perfect cocycle property for u, observe that Q has the cocycle property

$$Q(t_1+t_2,\omega) = Q\big(t_2,\theta(t_1,\omega)\big)Q(t_1,\omega),\ t_1,t_2 \geq 0,\ \omega \in \Omega. \tag{10}$$

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The cocycle property for u will follow from the identity

$$\begin{aligned} Q(t_1,\omega)[v(t_1+t_2,f,\omega) + T_{t_2}Z(t_1,\omega)] \\ = v\big(t_2, Q(t_1,\omega)[v(t_1,f,\omega)+Z(t_1,\omega)], \theta(t_1,\omega)\big) \end{aligned} \tag{11}$$

for $t_1, t_2 \geq 0, \omega \in \Omega,\ f \in H$. Above identity holds by uniqueness of the solution to the random NSE (9).

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$$\begin{aligned} &|u\big(t_2, f, \theta(t_1,\omega)\big)|_H \\ &= Q(t_2, \theta(t_1,\omega)|v\big(t_2, f, \theta(t_1,\omega)\big) + Z(t_2, \theta(t_1,\omega)|_H \\ &\le Q(t_2, \theta(t_1,\omega)[|f|_H + c(\omega)] \\ &\le [\rho + c(\omega)]\|Q\|_\infty \|Q^{-1}\|_\infty, \end{aligned} \tag{12}$$

where $\|Q^{-1}\|_\infty := \sup_{0 \le t \le 2a} \|Q^{-1}(t)\|$ and $E \log c < \infty$.

Proof of Theorem-contd

To prove the integrability estimate in the theorem: Let $0 \leq t_1, t_2 \leq a$ and $f \in H$ with $|f|_H \leq \rho$. Then by an a priori estimate on v:

$$\begin{aligned} &|u\big(t_2, f, \theta(t_1,\omega)\big)|_H \\ &= Q(t_2, \theta(t_1,\omega)|v\big(t_2, f, \theta(t_1,\omega)\big) + Z(t_2, \theta(t_1,\omega)|_H \\ &\leq Q(t_2, \theta(t_1,\omega)[|f|_H + c(\omega)] \\ &\leq [\rho + c(\omega)]\|Q\|_\infty \|Q^{-1}\|_\infty, \end{aligned} \tag{12}$$

where $\|Q^{-1}\|_\infty := \sup\limits_{0 \leq t \leq 2a} \|Q^{-1}(t)\|$ and $E \log c < \infty$.

Using a priori estimates on Dv, we obtain

Proof of Theorem-contd

$$\begin{aligned}
&\|Du\big(t_2, f, \theta(t_1,\omega)\big)\|_{L(H)} \\
&\qquad = Q(t_2, \theta(t_1,\omega)\|Dv\big(t_2, f, \theta(t_1,\omega)\big)\|_{L(H)} \\
&\qquad \leq c_1(\omega)\|Q\|_\infty\|Q^{-1}\|_\infty \exp\{c_2(\omega)|f|_H^2\} \qquad \textbf{(13)}
\end{aligned}$$

where

$$E\log c_1 < \infty, \qquad Ec_2 < \infty.$$

Proof of Theorem-contd

The above two estimates imply

$$E \log^+ \sup_{\substack{0 \le t_1, t_2 \le a \\ |f|_H \le \rho}} |u(t_2, f, \theta(t_1, \cdot))|_H + E \log^+ \sup_{\substack{0 \le t_1, t_2 \le a \\ |f|_H \le \rho}} \|Du(t_2, f, \theta(t_1, \cdot))\|_{L(H)} < \infty. \qquad (14)$$

Proof of Theorem-contd

That is:

$$E \log^{+} \sup_{0 \leq t_1, t_2 \leq a} \|u(t_2, \cdot, \theta(t_1, \cdot))\|_{C^1} < \infty$$

where $\| \cdot \|_{C^1}$ denotes the C^1 norm on the closed ball $B(0, \rho)$ in H, center 0 and radius ρ. $\square$

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is a stationary random point or equilibrium for the cocycle (u, θ) if

$$u(t, Y(\omega), \omega) = Y(\theta(t, \omega))$$

for all $t \in \mathbf{R}^+$, and $\omega \in \Omega$.

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Let $Y : \Omega \to H$ be a stationary random point for the cocycle (u, θ) of SNSE (3) with $E \log^+ |Y| < \infty$.

Then $(Du(t, Y(\omega), \omega), \theta(t, \omega))$ is a compact linear cocycle.

Hyperbolic equilibrium

Let $Y : \Omega \to H$ be a stationary random point for the cocycle (u, θ) of SNSE (3) with $E \log^+ |Y| < \infty$.

Then $\big(Du(t, Y(\omega), \omega), \theta(t, \omega)\big)$ is a compact linear cocycle.
The Oseledec-Ruelle operator is given by

$$\Lambda(\omega) := \lim_{t\to\infty} \Big\{ \big[Du(t, Y(\omega), \omega)\big]^* \circ \big[Du(t, Y(\omega), \omega)\big] \Big\}^{1/2t}$$

Limit exists in the uniform operator norm in $L(H)$ perfectly in $\omega \in \Omega$-(Ruelle-Oseledec theorem). [Ru]

Lyapunov exponents-contd

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The Oseledec-Ruelle operator is compact, self-adjoint and non-negative with fixed discrete spectrum

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The Lyapunov exponents

$$\{\cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$$

are values of the almost sure limit

$$\lim_{t\to\infty} \frac{1}{t} \log |Du(t, Y(\omega), \omega)(g)|_H, \;\; g \in H.$$

Ergodicity vs. hyperbolicity

The stationary point Y is said to be hyperbolic if $\lambda_i \neq 0$ for all $i \geq 1$ and $\lambda_1 > 0$.

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Next result gives necessary and sufficient conditions for hyperbolicity of the zero equilibrium $Y \equiv 0$.

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Proof:

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Proof: Use the formula

$$\lambda_n = \gamma - \mu_n - \frac{1}{2}\sum_{k=1}^{\infty} |\sigma_k|^2, \;\; n \geq 1$$

for the Lyapunov exponents of the linearized cocycle $(Du(t, 0, \omega), \theta(t, \omega))$.

Linearization: Spectral theorem

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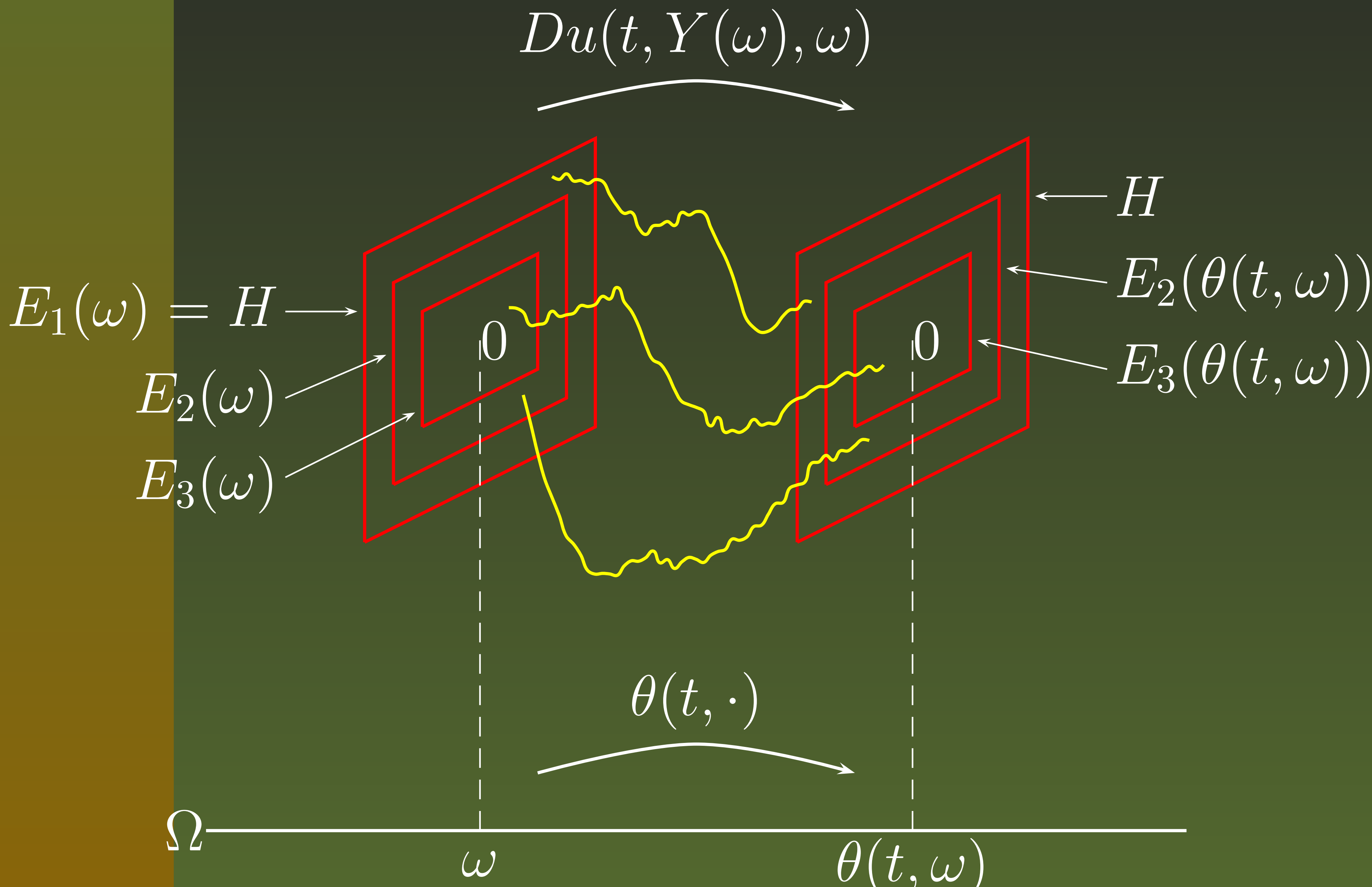

Random saddles

$\{\mathcal{U}(\omega), \mathcal{S}(\omega) : \omega \in \Omega\}$:= unstable and stable subspaces associated with the linearized cocycle $(Du(t, Y(\omega), \omega), \theta(t, \omega))$ ([Mo.3], [M.S]).

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Then get a measurable perfect invariant splitting

$$H = \mathcal{U}(\omega) \oplus \mathcal{S}(\omega),$$

$$Du(t, Y(\omega), \omega)(\mathcal{U}(\omega)) = \mathcal{U}(\theta(t, \omega)),$$

$$Du(t, Y(\omega), \omega)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t, \omega)),$$

for all $t \geq 0$.

Random saddles-Contd

Have exponential dichotomies:

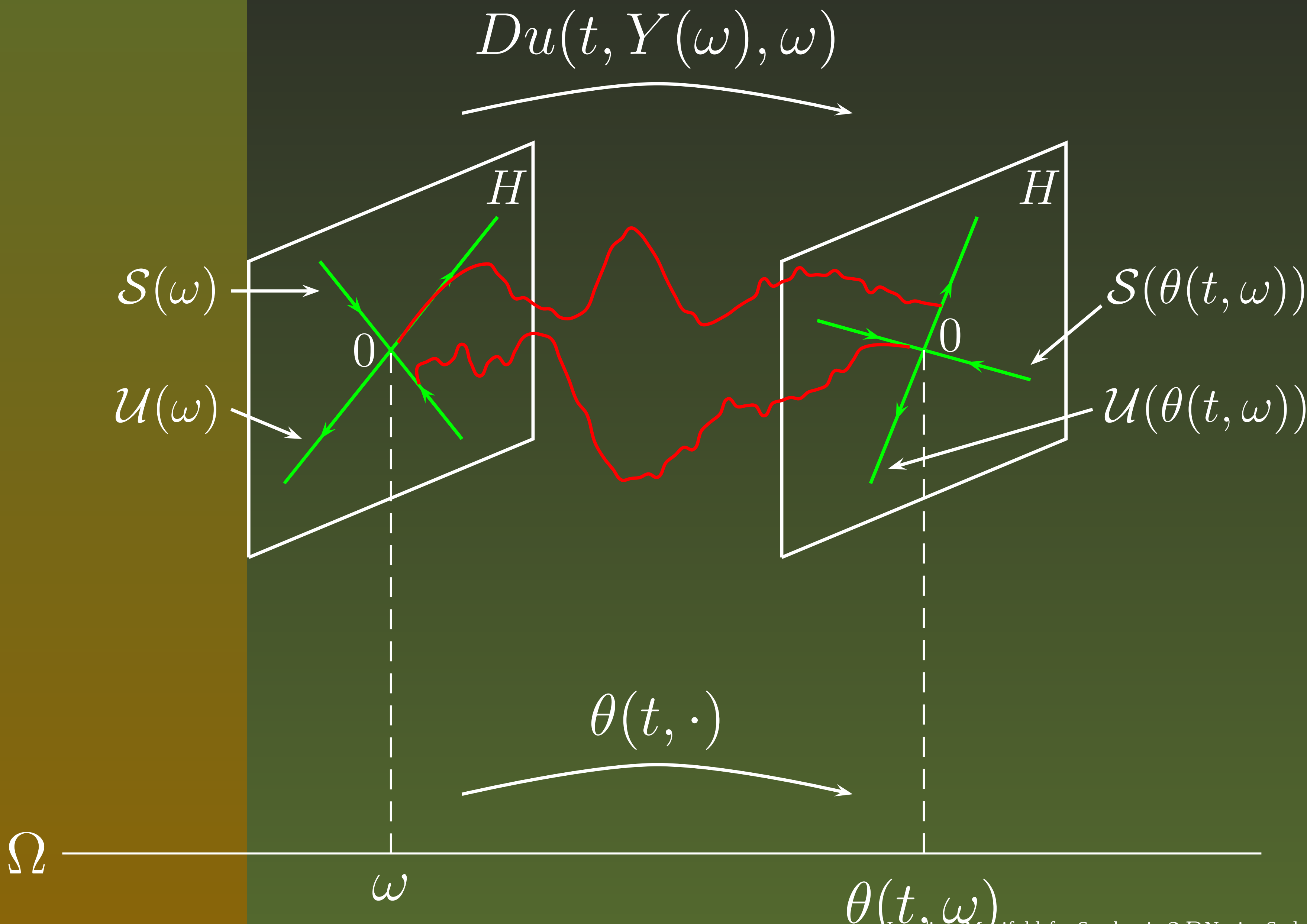
$$|Du(t, Y(\omega), \omega)(x)| \geq |x|e^{\delta_1 t}$$

for all $t \geq 0, x \in \mathcal{U}(\omega)$;

$$|Du(t, Y(\omega), \omega)(x)| \leq |x|e^{-\delta_2 t}$$

for all $t \geq 0, x \in \mathcal{S}(\omega)$, and $\delta_i > 0$, fixed, $i = 1, 2$.

Random saddles-contd



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First, we view a stationary tube around the hyperbolic equilibrium Y.

A stationary tube

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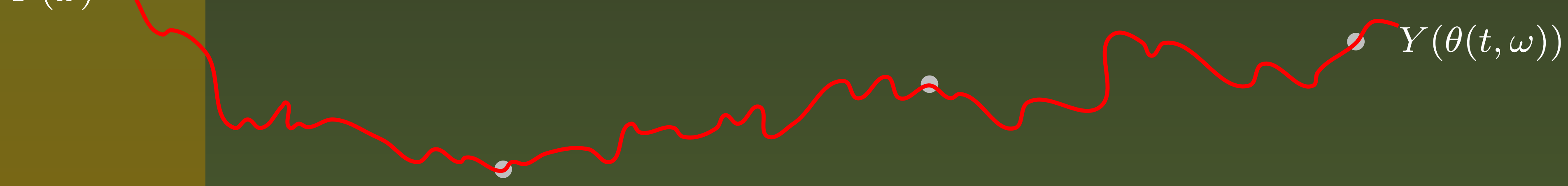
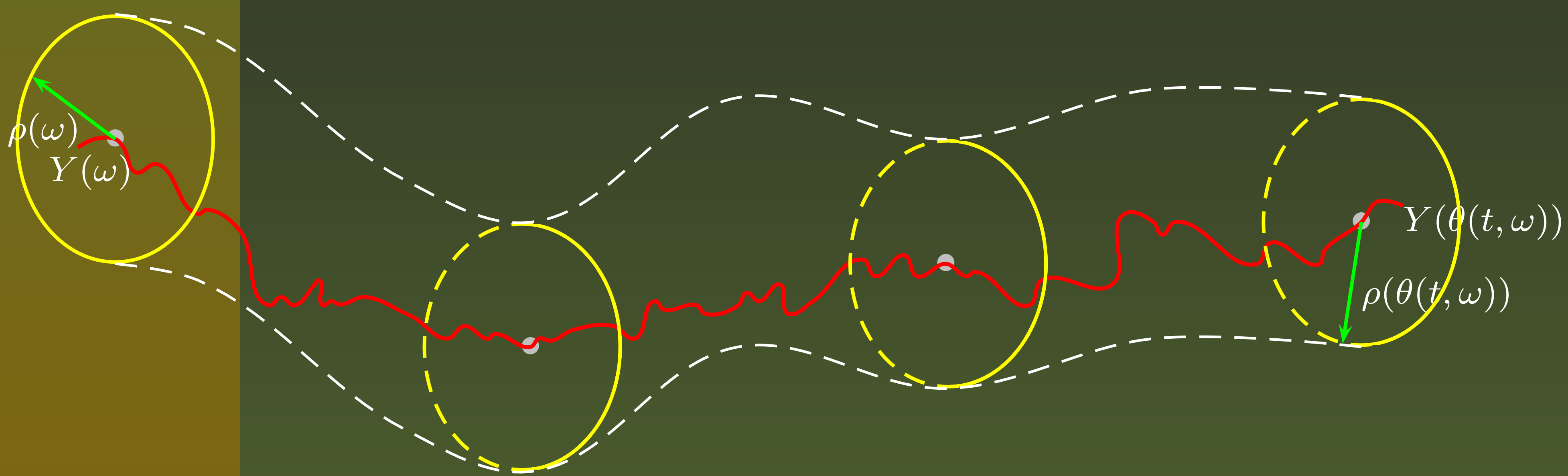


A stationary tube



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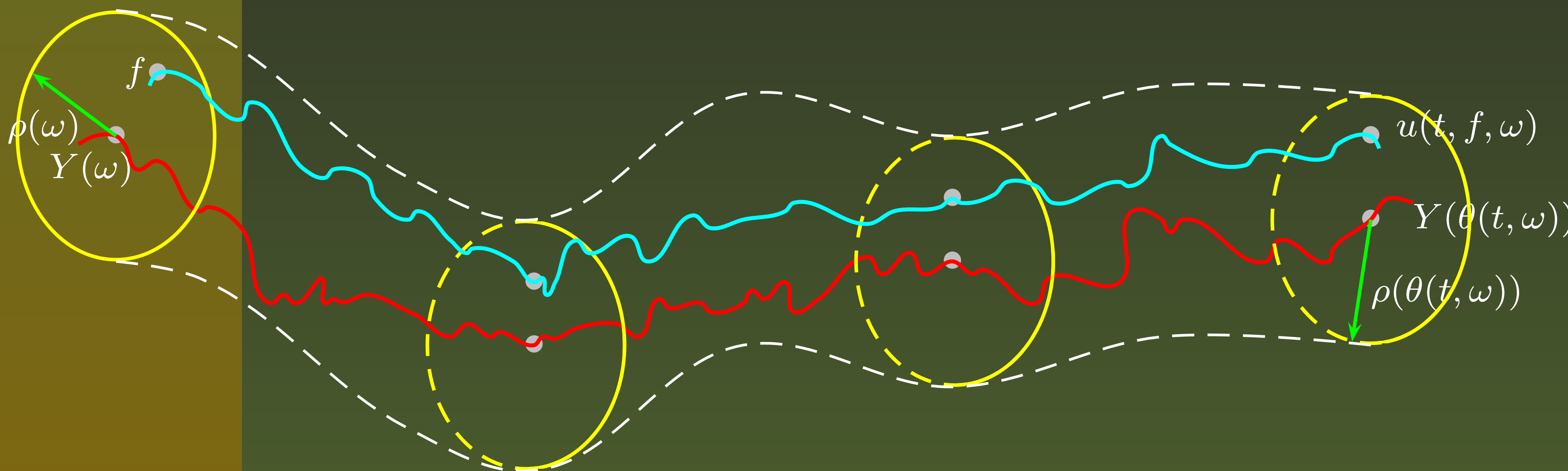
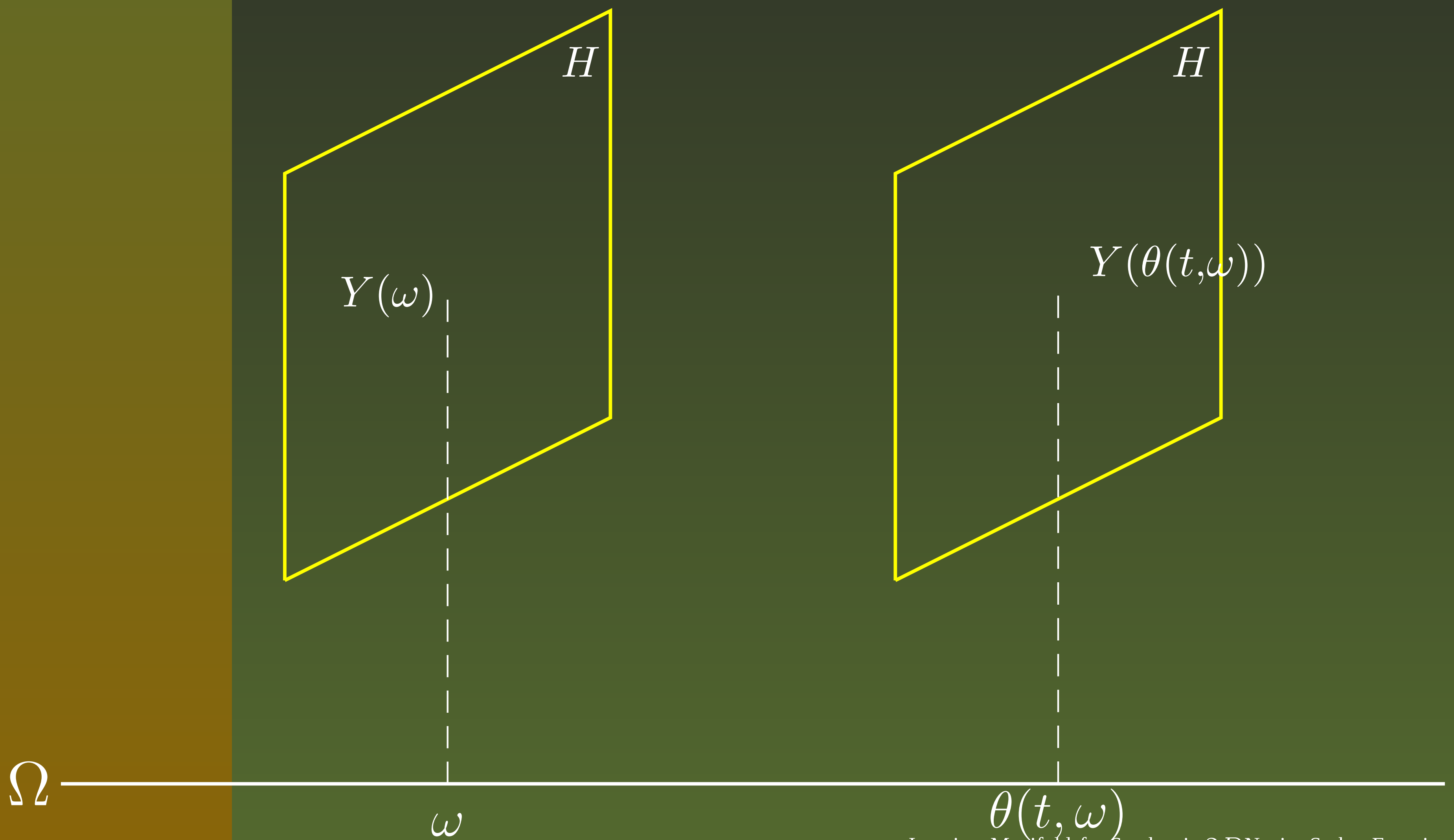


$$f \in H$$

Stable/unstable manifolds



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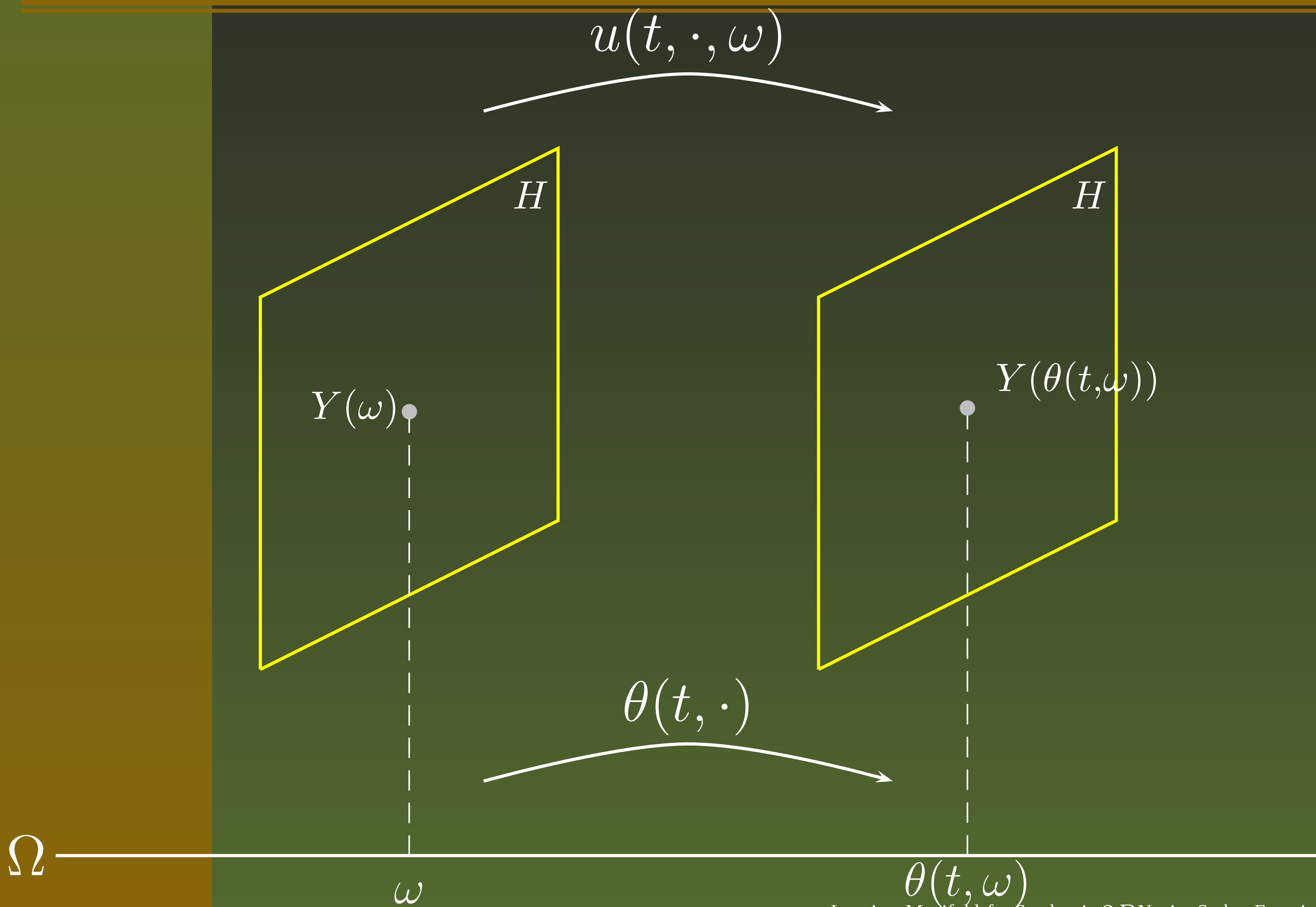

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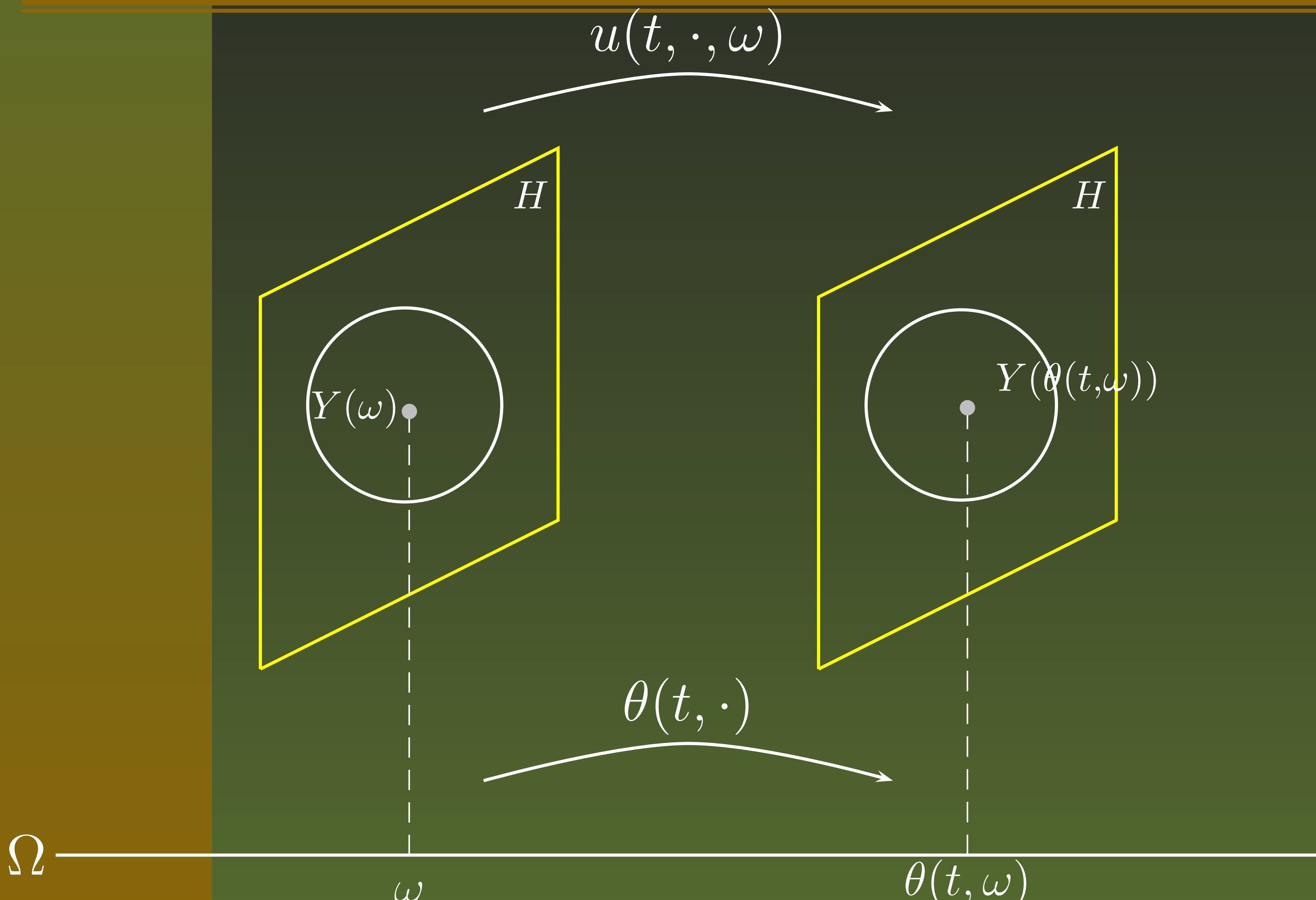

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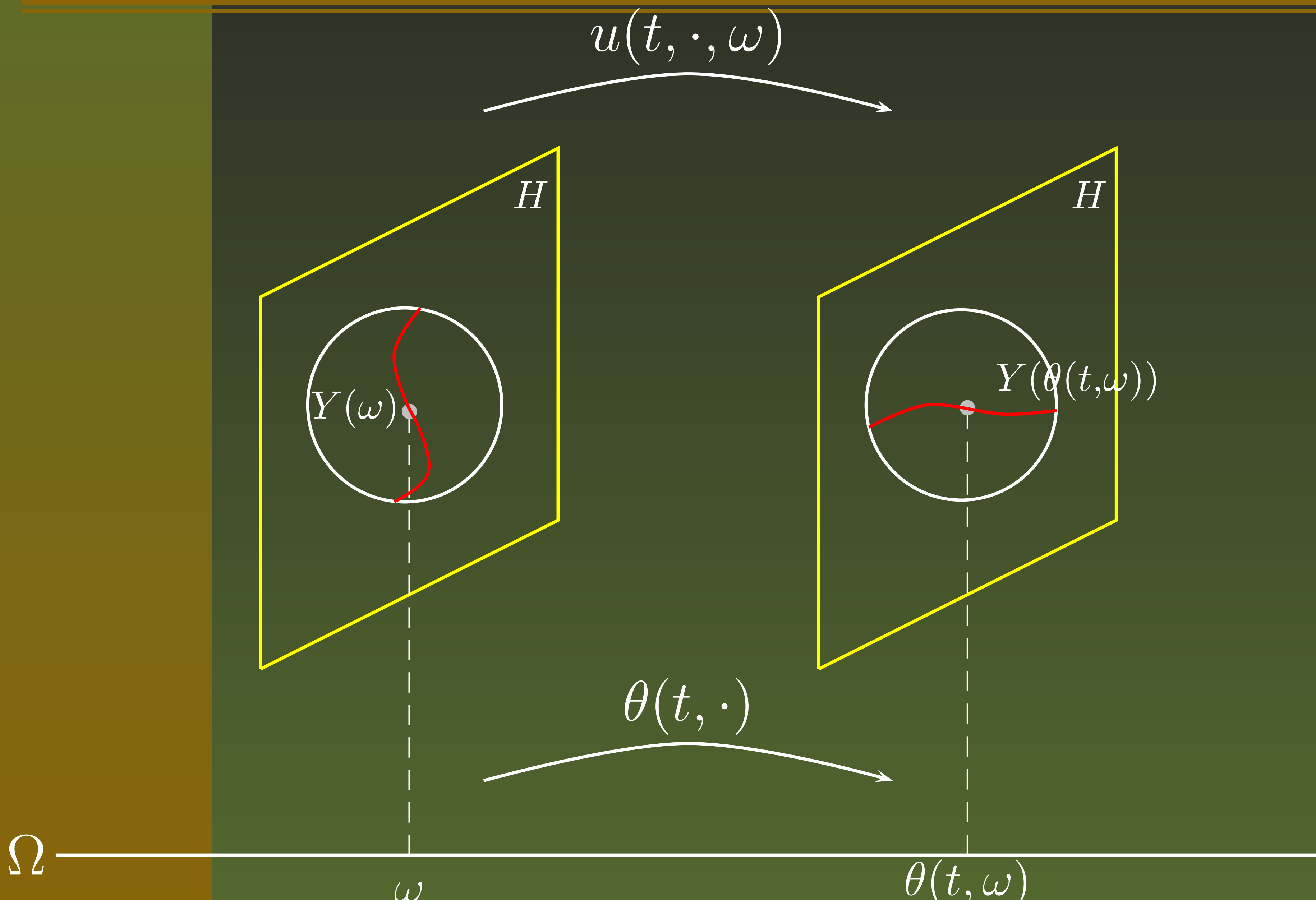

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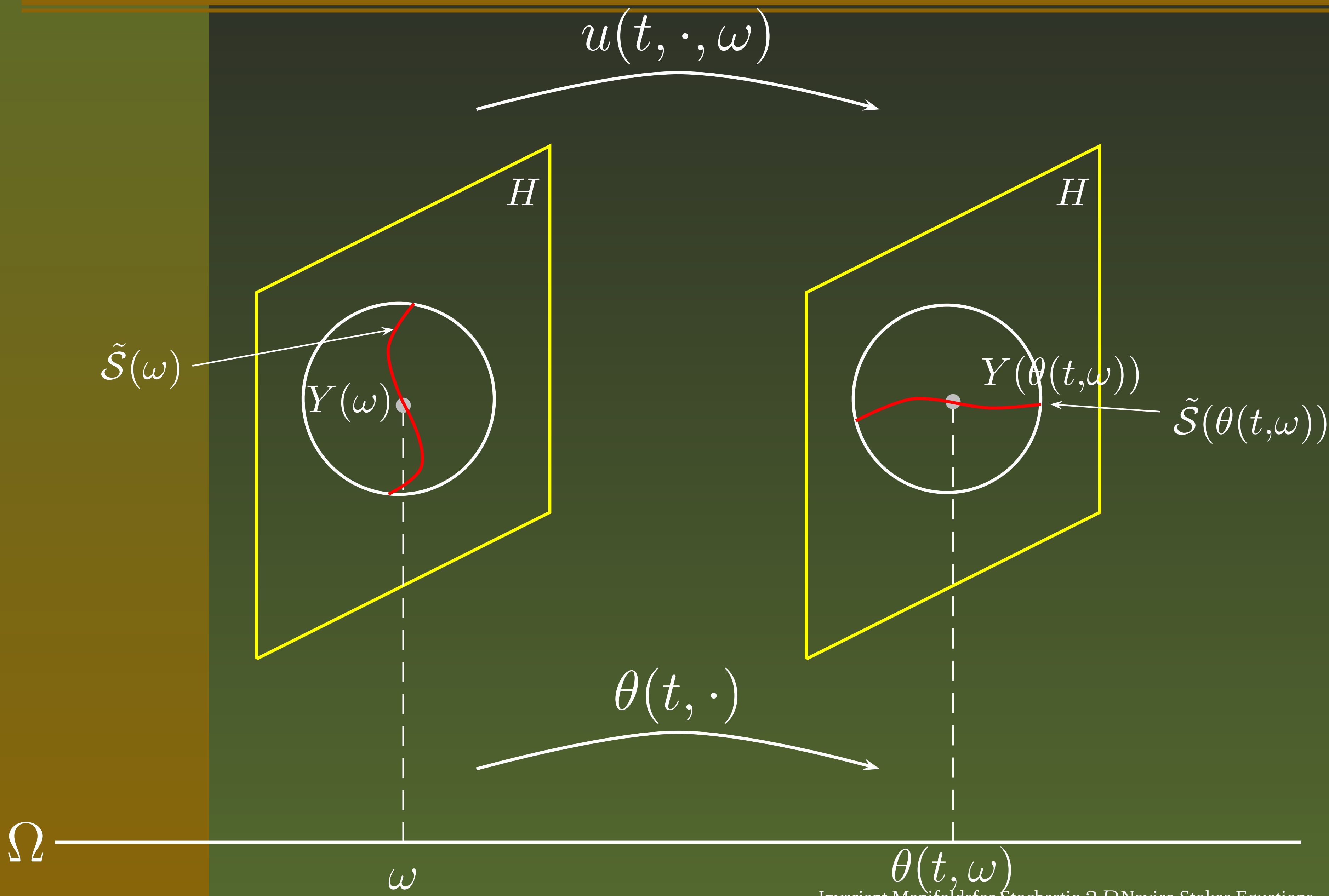

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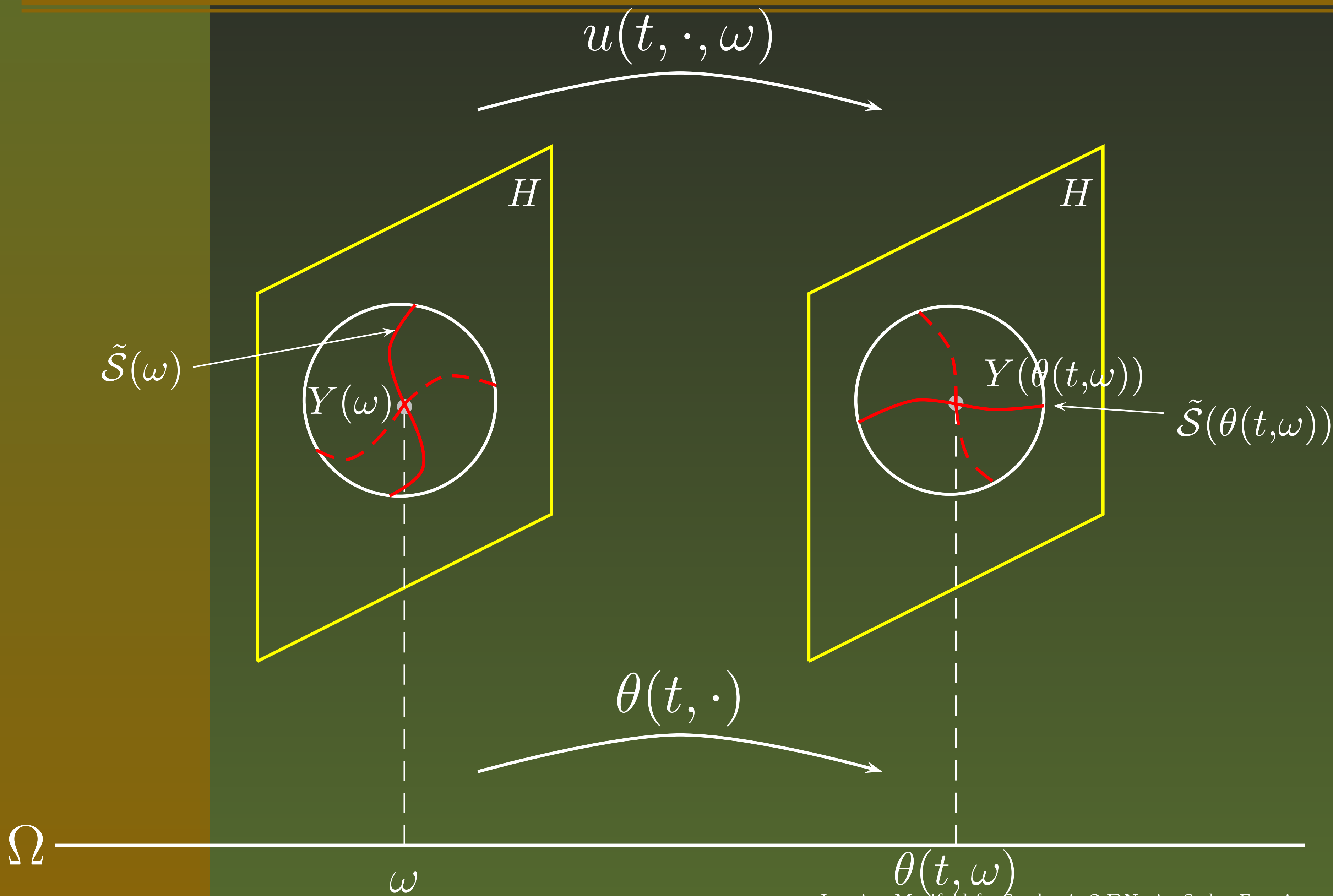

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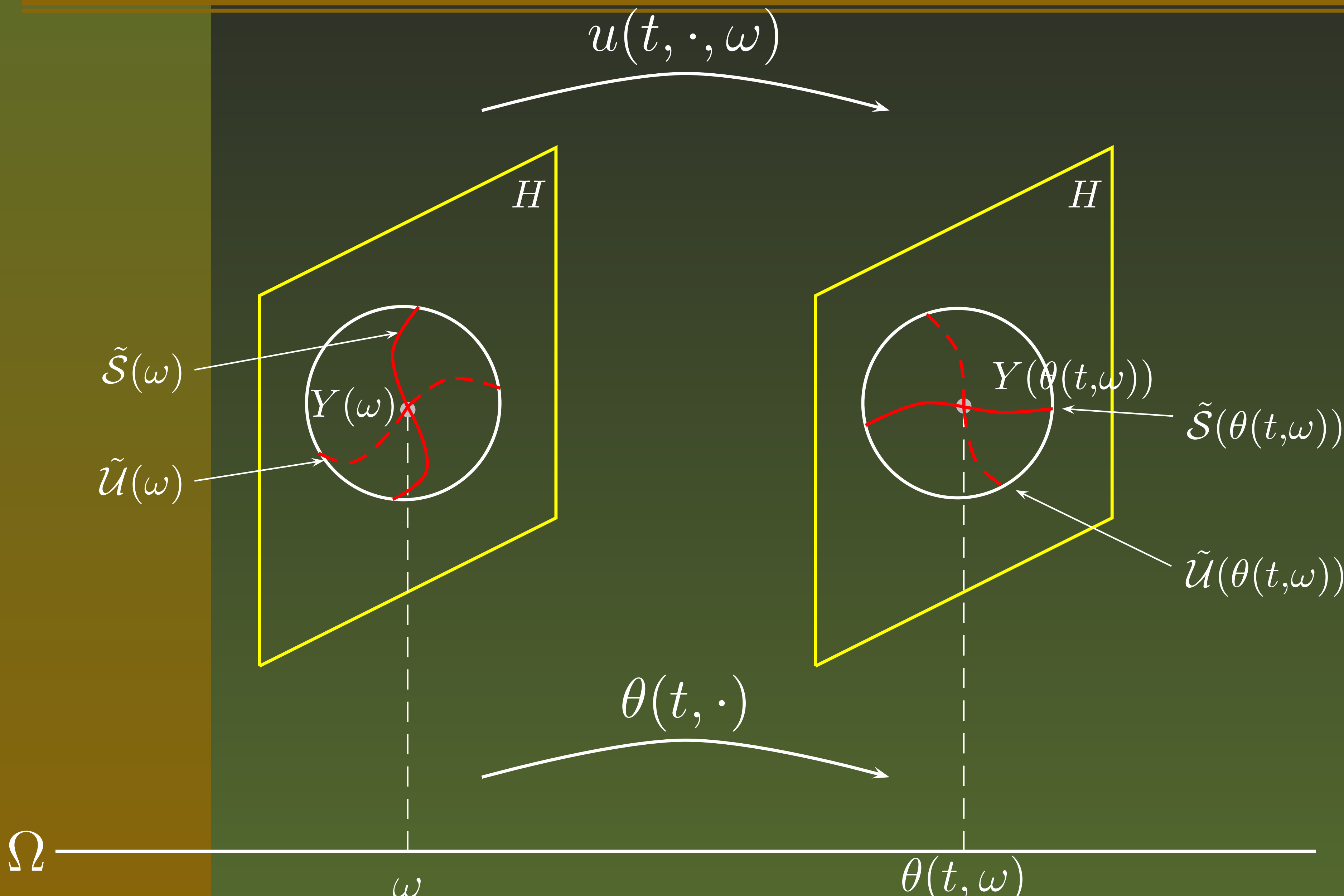

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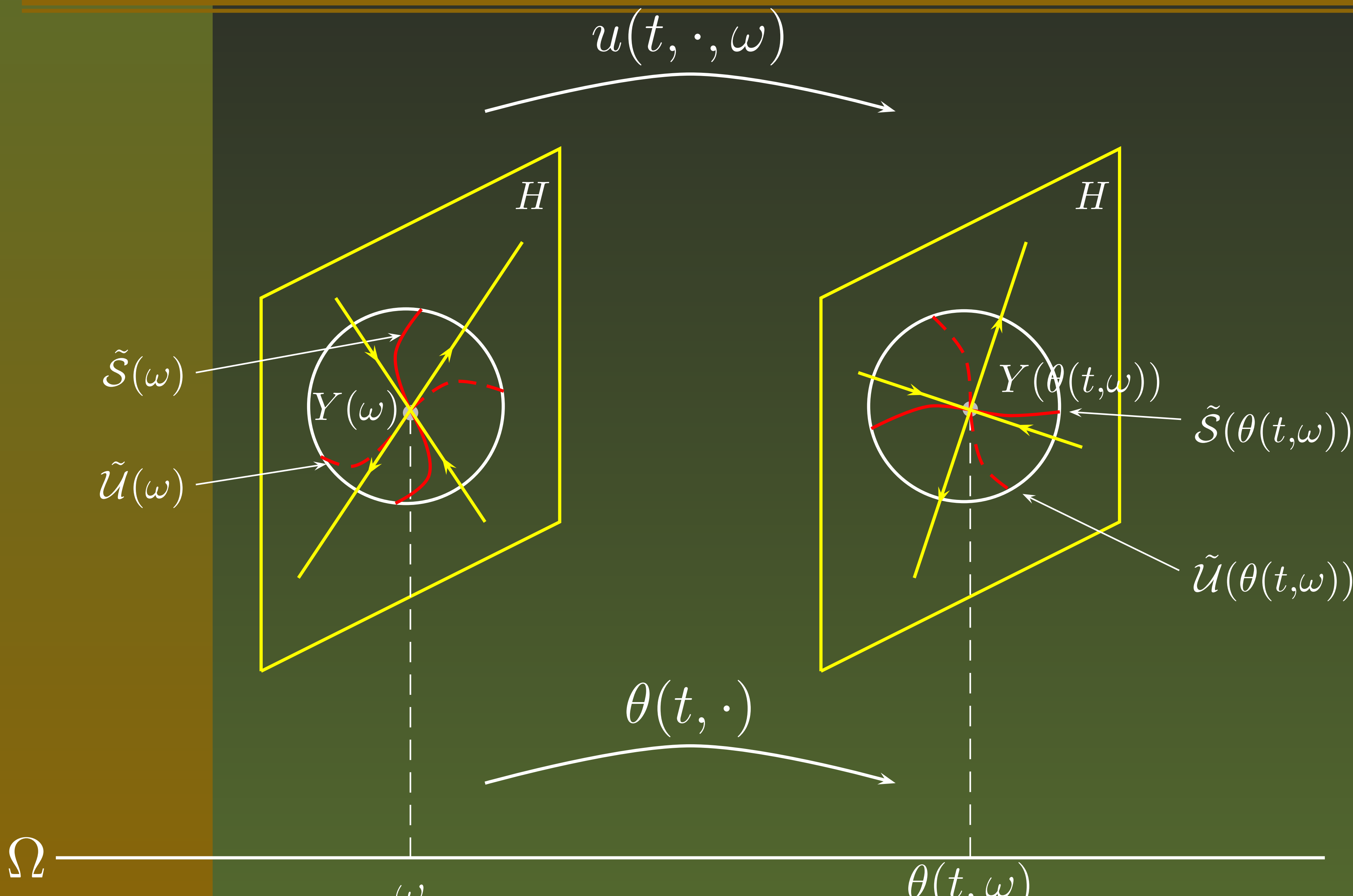

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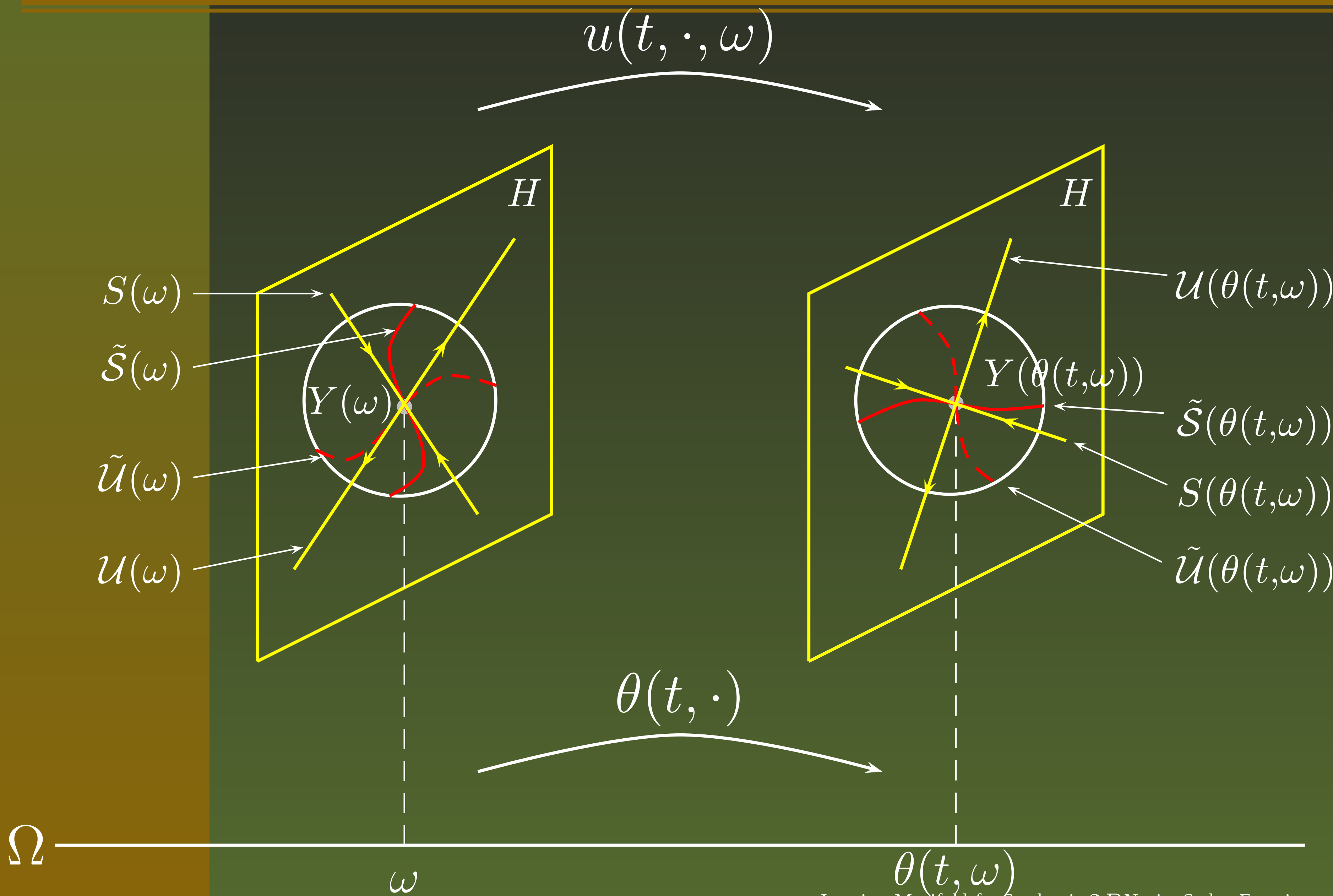

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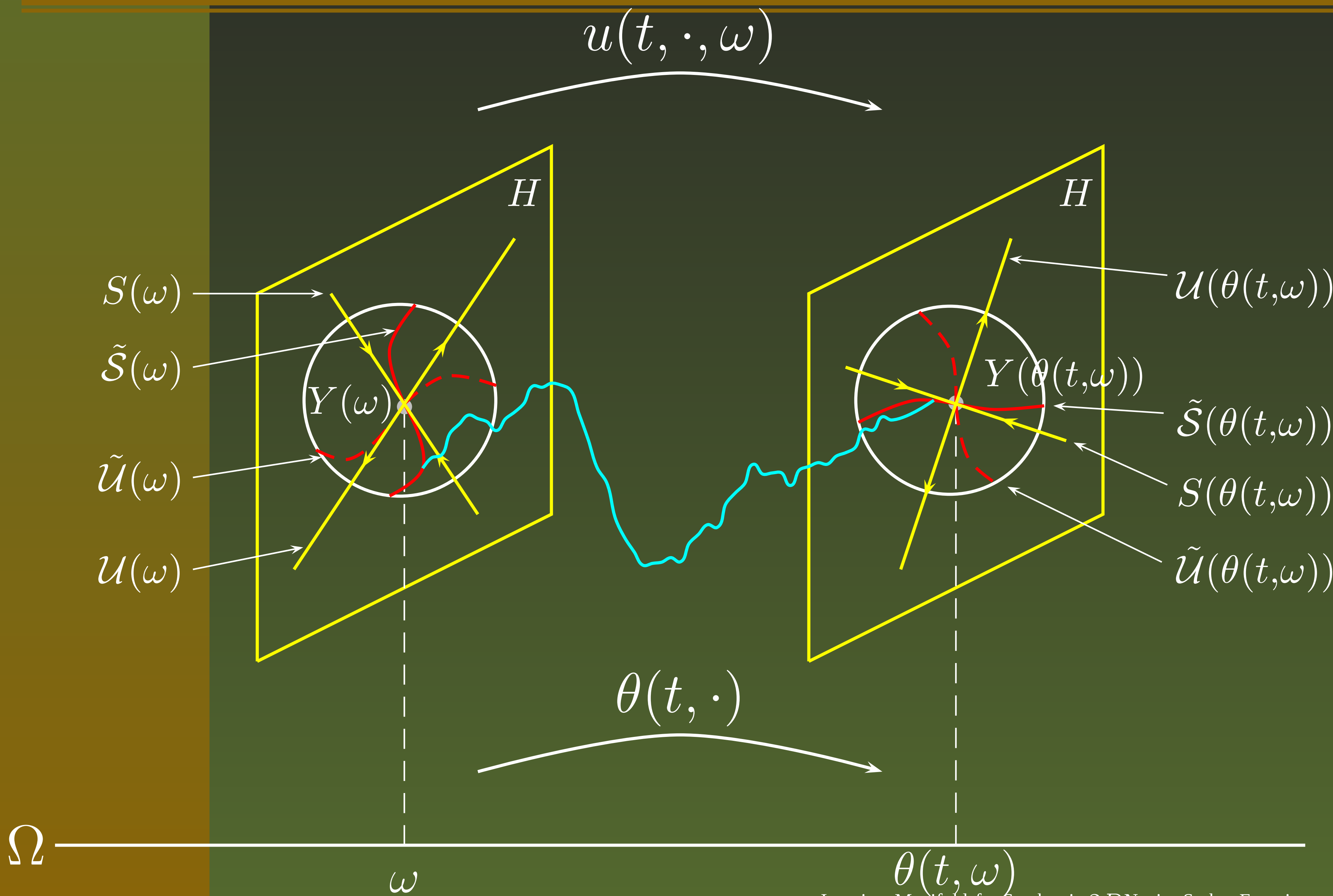

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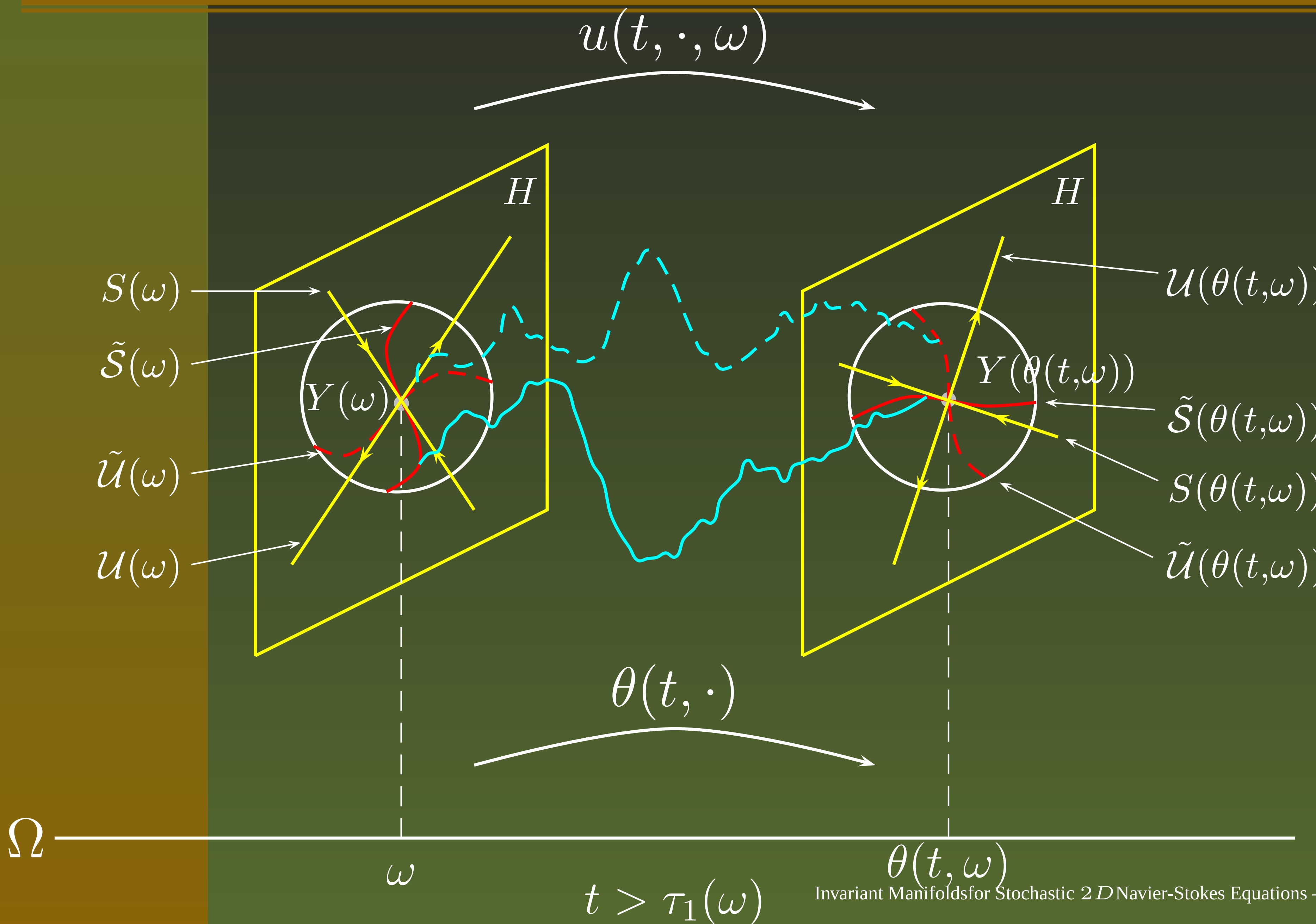

Invariant manifolds and foliations

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The local invariant manifold theorem characterizes the almost sure asymptotic stability of the random flow of the SNSE (3) in the neighborhood of an ergodic stationary point Y.

The global invariant foliation theorem gives random cocycle-invariant foliations in H, characterized by the Lyapunov exponents at an ergodic stationary point Y.

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Let (u, θ) be the cocycle generated by the SNSE (3) with $\gamma = 0$. Suppose Y is an ergodic stationary point of (3) with a Lyapunov spectrum $\{\lambda_i : i \geq 1\}$ and $\lambda_1 < 0$. Fix $\epsilon \in (0, -\lambda_1)$. Then there exist

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(i) a sure event $\Omega^* \in \mathcal{F}$ with $\theta(t, \cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbf{R}$,

(ii) $\mathcal{F}$-measurable random variables $\rho_i, \beta_i : \Omega^* \to (0, 1)$, $\beta_i > \rho_i \geq \rho_{i+1} > 0$, $i \geq 1$, such that for each $\omega \in \Omega^*$, the following is true:

Invariant manifolds-contd

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For each $i \geq 1$, there is a C^1 submanifold $\tilde{S}_i(\omega)$ of $B(Y(\omega), \rho_i(\omega))$ with the following properties:

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(a) $\tilde{\mathcal{S}}_i(\omega)$ is the set of all $f \in B(Y(\omega), \rho_i(\omega))$ such that

$$|u(n, f, \omega) - Y(\theta(n, \omega))|_H \leq \beta_i(\omega) \exp\{(\lambda_i + \epsilon)n\}$$

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for all integers $n \geq 0$.
Furthermore, for each $f \in \tilde{\mathcal{S}}_i(\omega)$:

$$\limsup_{t\to\infty} \frac{1}{t} \log |u(t, f, \omega) - Y(\theta(t, \omega))|_H \leq \lambda_i$$

Invariant manifolds-contd

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Each $\tilde{\mathcal{S}}_{i+1}(\omega)$ is a submanifold of $\tilde{\mathcal{S}}_i(\omega)$; and $T_{Y(\omega)}\tilde{\mathcal{S}}_i(\omega) = E_i(\omega)$. In particular, $\operatorname{codim} \tilde{\mathcal{S}}_i(\omega) = \operatorname{codim} E_i(\omega)$ (fixed and finite).

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(b) $$\limsup_{t\to\infty} \frac{1}{t} \log\left[\sup\left\{ \frac{|u(t,f_1,\omega) - u(t,f_2,\omega)|_H}{|f_1 - f_2|_H} : f_1 \neq f_2,\ f_1, f_2 \in \tilde{\mathcal{S}}_i(\omega) \right\}\right] \leq \lambda_i$$

Invariant manifolds-contd

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(c) (Cocycle-invariance):
There exist $\tau_i(\omega) \geq 0$ such that

$$u(t, \cdot, \omega)(\tilde{\mathcal{S}}_i(\omega)) \subseteq \tilde{\mathcal{S}}_i(\theta(t, \omega))$$

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for all $t \geq \tau_i(\omega)$. Also

$$Du(t, Y(\omega), \omega)(E_i(\omega)) \subseteq E_i(\theta(t, \omega)), \quad t \geq 0.$$

Global invariant flag theorem

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Define the random sets $M_i(\omega)$, $\omega \in \Omega^*$, $i \geq 1$, by

$$M_i(\omega)$$

$$:= \left\{ f \in H : \overline{\lim_{t\to\infty}} \frac{1}{t} \log |u(t, f, \omega) - Y(\theta(t, \omega)|_H \leq \lambda_i \right\}$$

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for $i \geq 1$, $\omega \in \Omega^*$.

For fixed $i \geq 1, \omega \in \Omega^*$, define the sequence $\{S_i^n(\omega)\}_{n=1}^{\infty}$, inductively by:

Global invariant flag-contd

$$S_i^1(\omega) := \tilde{\mathcal{S}}_i(\omega)$$

$$S_i^n(\omega) := \begin{cases} u(n,\cdot,\omega)^{-1}\big[\tilde{\mathcal{S}}_i\big(\theta(n,\omega)\big)\big], \\ \qquad\qquad \text{if } S_i^{n-1}(\omega) \subseteq u(n,\cdot,\omega)^{-1}\big[\tilde{\mathcal{S}}_i\big(\theta(n,\omega)\big)\big] \\ S_i^{n-1}(\omega), \qquad\qquad \text{otherwise,} \end{cases}$$

for all $n \geq 2$, where $\tilde{\mathcal{S}}_i(\omega), i \geq 1$, are the local invariant C^1 manifolds at $Y(\omega)$.

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for all $n \geq 2$, where $\tilde{\mathcal{S}}_i(\omega), i \geq 1$, are the local invariant C^1 manifolds at $Y(\omega)$.

Then the following is true for each $i \geq 1$ and $\omega \in \Omega^*$:

Global invariant flag-contd

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(i) Each $M_i(\omega)$ is cocycle- invariant:

$$u(t,\cdot,\omega)\big(M_i(\omega)\big) \subseteq M_i\big(\theta(t,\omega)\big)$$

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(ii) $S_i^n(\omega) \subseteq S_i^{n+1}(\omega)$ for all $n \geq 1$, and

$$M_i(\omega) = \bigcup_{n=1}^{\infty} S_i^n(\omega)$$

(perfectly in ω).

Global invariant flag-contd

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(iii) $M_{i+1}(\omega) \subseteq M_i(\omega)$.

Global invariant flag-contd

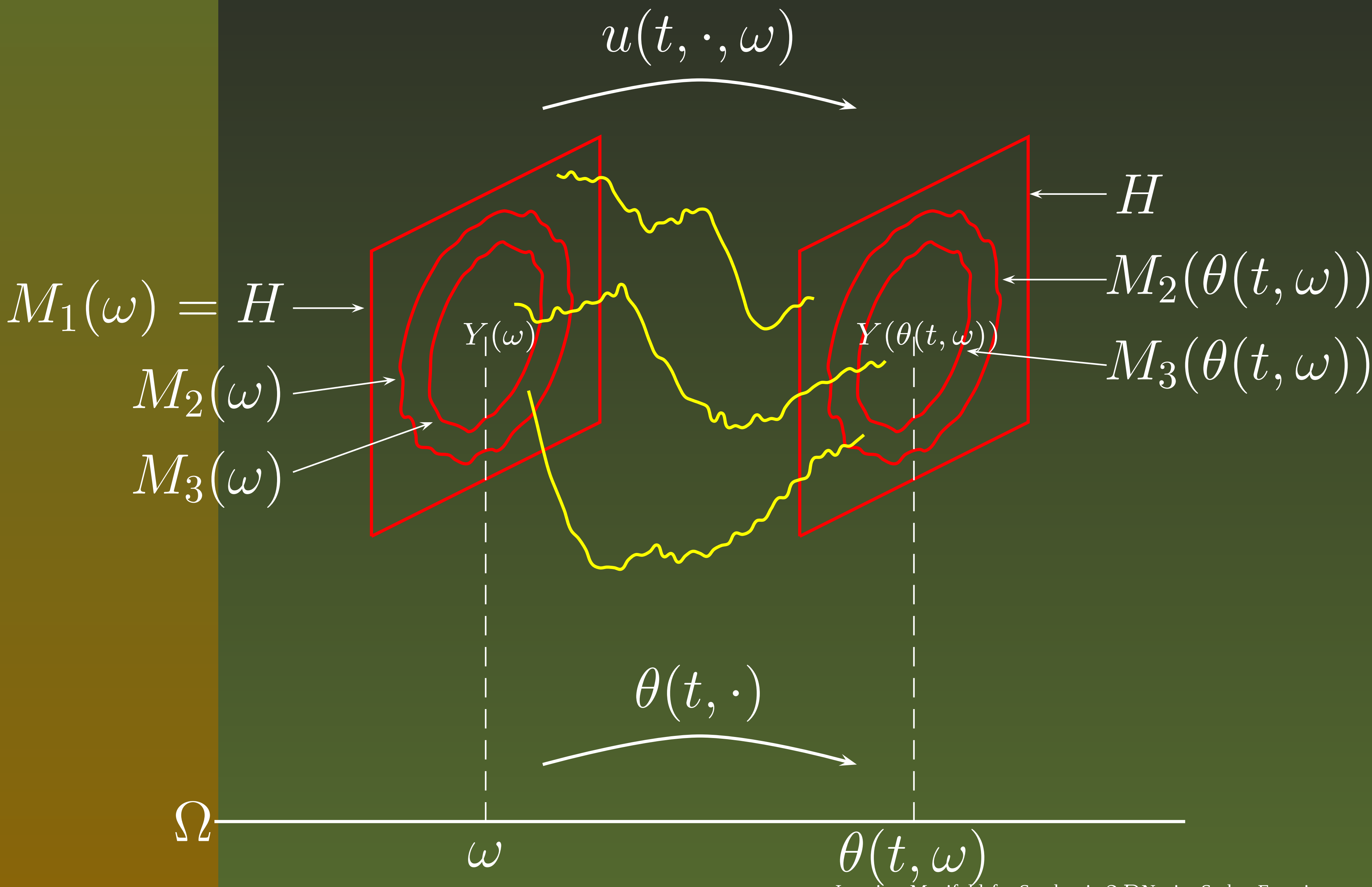
(iii) $M_{i+1}(\omega) \subseteq M_i(\omega)$.

(iv) For any $f \in M_i(\omega) \backslash M_{i+1}(\omega)$,

$$\overline{\lim_{t\to\infty}} \quad \frac{1}{t} \log |u(t, f, \omega) - Y(\theta(t, \omega))|_H \in (\lambda_{i+1}, \lambda_i] \,.$$

Global Invariant Flag

Global Invariant Flag



Burgers spde

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Similar (C^∞) dynamics holds for one-dimensional Burgers equation with affine white noise:

Burgers spde

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$$\left.\begin{aligned} du(t) &= \nu\Delta u\,dt - u\frac{\partial u}{\partial \xi}\,dt + \gamma u(t)\,dt + \sigma_0\,dW_0(t) \\ &\qquad + \sigma u(t)\,dW(t),\ t>0,\ \ \xi\in[0,1], \\ u(t,0) &= u(t,1) = 0 \ \text{ for all } t>0, \\ u(0,\xi) &= f(\xi),\ \xi\in[0,1]. \end{aligned}\right\}$$

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THANK YOU!